

COMBINATORIAL AND MODEL-THEORETICAL PRINCIPLES RELATED TO REGULARITY OF ULTRAFILTERS AND COMPACTNESS OF TOPOLOGICAL SPACES. III.

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ABSTRACT. We generalize the results from [L2]; in particular the present results apply to singular cardinals, too.

See [L4, KV, HNV] for definitions and notation.
We shall need the following theorem from [L5].

Theorem 1. *If λ is a singular cardinal, then an ultrafilter is (λ, λ) -regular if and only if it is either $(\text{cf } \lambda, \text{cf } \lambda)$ -regular or (λ^+, λ^+) -regular.*

Corollary 2. *Suppose that λ is a singular cardinal, and consider the topological space X obtained by forming the disjoint union of the topological spaces λ^+ and $\text{cf } \lambda$, both endowed with the order topology.*

Then, for every ultrafilter D , the space X is D -compact if and only if D is not (λ, λ) -regular.

Thus, X is productively $[\lambda', \mu']$ -compact if and only if there exists a (λ', μ') -regular not (λ, λ) -regular ultrafilter. In particular, X is not productively $[\lambda, \lambda]$ -compact.

Proof. By Theorem 1, D is not (λ, λ) -regular if and only if it is neither $(\text{cf } \lambda, \text{cf } \lambda)$ -regular nor (λ^+, λ^+) -regular.

Hence, by [L2, Proposition 1], and since both λ^+ and $\text{cf } \lambda$ are regular cardinals, D is not (λ, λ) -regular if and only if both λ^+ and $\text{cf } \lambda$ are D -compact. This is clearly equivalent to X being D -compact.

The last statement is immediate from [C2, Theorem 1.7], also stated in [L2, Theorem 2]. □

2000 *Mathematics Subject Classification.* Primary 03E05, 54B10, 54D20, 54A20; Secondary 03E75.

Key words and phrases. Regular ultrafilters; compactness of products of topological spaces.

The author has received support from MPI and GNSAGA. We wish to express our gratitude to X. Caicedo for stimulating discussions and correspondence.

Let $\mathbf{2} = \{0, 1\}$ denote the two-elements topological space with the discrete topology. If $\lambda \leq \mu$ are cardinals, let $\mathbf{2}^\mu$ be the Tychonoff product of μ -many copies of $\mathbf{2}$, and let $\mathbf{2}_\lambda^\mu$ denote the subset of $\mathbf{2}^\mu$ consisting of all those functions $h : \mu \rightarrow \mathbf{2}$ such that $|\{\alpha \in \mu \mid h(\alpha) = 1\}| < \lambda$.

In passing, let us mention that, when $\mu = \aleph_\omega$, the space $\mathbf{2}_\mu^\mu$ provides an example of a linearly Lindelöf not Lindelöf space. See [AB, Example 4.1]. Compare also [S, Example 4.2].

Notice that $\mathbf{2}_\lambda^\mu$ is a Tychonoff topological group with a base of clopen sets.

Set theoretically, $\mathbf{2}_\lambda^\mu$ is in a one to one correspondence (via characteristic functions) with $S_\lambda(\mu)$, the set of all subsets of μ of cardinality $< \lambda$. Since many properties of ultrafilters are defined in terms of $S_\lambda(\mu)$, for sake of convenience, in what follows we shall deal with $S_\lambda(\mu)$, rather than $\mathbf{2}_\lambda^\mu$. Henceforth, we shall deal with the topology induced on $S_\lambda(\mu)$ by the above correspondence.

In detail, $S_\lambda(\mu)$ is endowed with the smallest topology containing, as open sets, all sets of the form $X_\alpha = \{x \in S_\lambda(\mu) \mid \alpha \in x\}$ (α varying in μ), as well as their complements. Thus, a base for the topology consists of all finite intersections of the above sets; that is, the elements of the base are the sets $\{x \in S_\lambda(\mu) \mid \alpha_1 \in x, \alpha_2 \in x, \dots, \alpha_n \in x, \beta_1 \notin x, \beta_2 \notin x, \dots, \beta_m \notin x\}$, with n, m varying in ω and $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m$ varying μ .

Notice that this topology is finer than the topology on $S_\lambda(\mu)$ used in [L2].

With the above topology, $S_\lambda(\mu)$ and $\mathbf{2}_\lambda^\mu$ are homeomorphic, thus $S_\lambda(\mu)$ can be given the structure of a Tychonoff topological group.

Notice that if $\lambda \leq \mu$ then $S_\lambda(\mu)$ is not $[\lambda, \lambda]$ -compact. Indeed, for $\alpha \in \mu$, let $Y_\alpha = \{x \in S_\lambda(\mu) \mid \alpha \notin x\}$. If $Z \subseteq \mu$ and $|Z| = \lambda$ then $(Y_\alpha)_{\alpha \in Z}$ is an open cover of $S_\lambda(\mu)$ by λ -many sets, $< \lambda$ of which never cover $S_\lambda(\mu)$.

Proposition 3. *For every ultrafilter D and every cardinal λ , the topological space $S_\lambda(\lambda)$ is D -compact if and only if D is not (λ, λ) -regular.*

Proof. Suppose that D is an ultrafilter over I and that $S_\lambda(\lambda)$ is D -compact. For every $f : I \rightarrow S_\lambda(\lambda)$ there exists $x \in S_\lambda(\lambda)$ such that $f(i)_{i \in I}$ D -converges to x . If $\alpha \in \lambda$ and $\{i \in I \mid \alpha \in f(i)\} \in D$ then $\alpha \in x$, since otherwise $Y = \{z \in S_\lambda(\lambda) \mid \alpha \notin z\}$ is an open set containing x , and $\{i \in I \mid f(i) \in Y\} = \{i \in I \mid \alpha \notin f(i)\} \notin D$, contradicting D -convergence.

Whence, $\{\alpha \in \lambda \mid \{i \in I \mid \alpha \in f(i)\} \in D\} \subseteq x \in S_\lambda(\lambda)$, and thus x has cardinality $< \lambda$; that is, f does not witness (λ, λ) -regularity of D . Since f has been chosen arbitrarily, D is not (λ, λ) -regular.

Conversely, suppose that D over I is not (λ, λ) -regular, and let $f : I \rightarrow S_\lambda(\lambda)$. Then $x = \{\alpha \in \lambda \mid \{i \in I \mid \alpha \in f(i)\} \in D\}$ has cardinality $< \lambda$ and hence is in $S_\lambda(\lambda)$. We show that f D -converges to x . Indeed, let Y be a neighborhood of x : we have to show that $\{i \in I \mid f(i) \in Y\} \in D$. Without loss of generality, we can suppose that Y is an element of the base of $S_\lambda(\lambda)$, that is, Y has the form $\{z \in S_\lambda(\lambda) \mid \alpha_1 \in z, \alpha_2 \in z, \dots, \alpha_n \in z, \beta_1 \notin z, \beta_2 \notin z, \dots, \beta_m \notin z\}$. Since D is closed under finite intersections, then $\{i \in I \mid f(i) \in Y\} \in D$ if and only if $\{i \in I \mid \alpha_1 \in f(i)\} \in D$ and $\{i \in I \mid \alpha_2 \in f(i)\} \in D$ and... and $\{i \in I \mid \alpha_n \in f(i)\} \in D$ and $\{i \in I \mid \beta_1 \notin f(i)\} \in D$ and... and $\{i \in I \mid \beta_m \notin f(i)\} \in D$. But all the above sets are actually in D , by the definition of x and since $x \in Y$ and D is an ultrafilter; thus f D -converges to x .

Since f was arbitrary, every $f : I \rightarrow S_\lambda(\lambda)$ D -converges, and thus $S_\lambda(\lambda)$ is D -compact. \square

Corollary 4. *The space $S_\lambda(\lambda)$ is productively $[\lambda', \mu']$ -compact if and only if there exists a (λ', μ') -regular not- (λ, λ) -regular ultrafilter.*

Proof. Immediate from Proposition 3 and [C2, Theorem 1.7]. \square

In the statements of the next theorems the word “productively”, when included within parentheses, can be equivalently inserted or omitted.

Theorem 5. *For all infinite cardinals λ, μ, κ , the following are equivalent:*

- (i) *Every productively $[\lambda, \mu]$ -compact topological space is (productively) $[\kappa, \kappa]$ -compact.*
- (ii) *Every productively $[\lambda, \mu]$ -compact family of topological spaces is productively $[\kappa, \kappa]$ -compact.*
- (iii) *Every (λ, μ) -regular ultrafilter is (κ, κ) -regular.*
- (iv) *Every productively $[\lambda, \mu]$ -compact Hausdorff normal topological space with a base of clopen sets is productively $[\kappa, \kappa]$ -compact.*
- (v) *Every productively $[\lambda, \mu]$ -compact Tychonoff topological group with a base of clopen sets is (productively) $[\kappa, \kappa]$ -compact.*

If κ is regular, then the preceding conditions are also equivalent to:

- (vi) *Every productively $[\lambda, \mu]$ -compact Hausdorff normal topological space with a base of clopen sets is $[\kappa, \kappa]$ -compact.*

Proof. Let us denote by (i)_p Condition (i) when the second occurrence of the word “productively” is included, and simply by (i) when it is omitted. Similarly, for condition (v).

The equivalence of (i)-(iii) has been proved in [L2, Theorem 1], where it has also been proved that, for κ regular, they are equivalent to (vi).

Since (ii) \Rightarrow (i)_p \Rightarrow (i) are trivial, we get that (i), (ii), (iii), (i)_p are all equivalent, and equivalent to (vi) for κ regular.

(ii) \Rightarrow (iv) and (ii) \Rightarrow (v)_p \Rightarrow (v) are trivial.

If (iii) fails, then there is a (λ, μ) -regular ultrafilter which is not (κ, κ) -regular, thus, for κ singular, the space X of Corollary 2 is productively $[\lambda, \mu]$ -compact. For κ regular, take $X = \kappa$ with the order topology (see [L2]). X is Hausdorff, normal, with a base of clopen sets, but not productively $[\kappa, \kappa]$ -compact, again by Corollary 2, thus (iv) fails. We have proved (iv) \Rightarrow (iii).

(v) \Rightarrow (iii) is similar, using Corollary 4, since $S_\kappa(\kappa)$ is not $[\kappa, \kappa]$ -compact. \square

Theorem 6. *For all infinite cardinals λ, μ , and for any family $(\kappa_i)_{i \in I}$ of infinite cardinals, the following are equivalent:*

(i) *Every productively $[\lambda, \mu]$ -compact topological space is (productively) $[\kappa_i, \kappa_i]$ -compact for some $i \in I$.*

(ii) *Every productively $[\lambda, \mu]$ -compact family of topological spaces is productively $[\kappa_i, \kappa_i]$ -compact for some $i \in I$.*

(iii) *Every (λ, μ) -regular ultrafilter is (κ_i, κ_i) -regular for some $i \in I$.*

(iv) *Every productively $[\lambda, \mu]$ -compact Hausdorff normal topological space with a base of clopen sets is productively $[\kappa_i, \kappa_i]$ -compact for some $i \in I$.*

(v) *Every productively $[\lambda, \mu]$ -compact Tychonoff topological group with a base of clopen sets is (productively) $[\kappa_i, \kappa_i]$ -compact for some $i \in I$.*

If every κ_i is regular, then the preceding conditions are also equivalent to:

(vi) *Every productively $[\lambda, \mu]$ -compact Hausdorff normal topological space with a base of clopen sets is $[\kappa_i, \kappa_i]$ -compact for some $i \in I$.*

Proof. The equivalence of (i)-(iii) has been proved in [L2, Theorem 3], thus, arguing as in the proof of Theorem 5, we get that (i), (ii), (iii), (i)_p are all equivalent.

(ii) \Rightarrow (iv) \Rightarrow (vi) and (ii) \Rightarrow (v)_p \Rightarrow (v) are trivial.

If (iii) fails, then there is a (λ, μ) -regular ultrafilter D which for no $i \in I$ is (κ_i, κ_i) -regular. By Proposition 3, for every $i \in I$ the topological space $S_{\kappa_i}(\kappa_i)$ is D -compact. Hence $X = \prod_{i \in I} S_{\kappa_i}(\kappa_i)$ is D -compact, thus productively $[\lambda, \mu]$ -compact, by [C2, Theorem 1.7]. However, X is a Tychonoff topological group with a base of clopen sets which for no $i \in I$ is $[\kappa_i, \kappa_i]$ -compact, thus (v) fails. We have proved (v) \Rightarrow (iii).

The proofs of (iv) \Rightarrow (iii) and (vi) \Rightarrow (iii) are similar, using the next proposition. If (iii) fails, then there is a (λ, μ) -regular ultrafilter D which for no $i \in I$ is (κ_i, κ_i) -regular. By the proof of Theorem 5, for

every $i \in I$ we have a D -compact topological space X_i which falsify 5(iv), resp., 5(vi). Then the space $X = \{x\} \dot{\cup} \bigcup_{i \in I} X_i$ we shall construct in the next definition is D -compact, thus productively $[\lambda, \mu]$ -compact, by [C2, Theorem 1.7], and makes (iv), resp., (vi), fail. \square

Definition 7. Given a family $(X_i)_{i \in I}$ of topological spaces, construct their *Frechet disjoint union* $X = \{x\} \dot{\cup} \bigcup_{i \in I} X_i$ as follows.

Set theoretically, X is the union of (disjoint copies) of the X_i 's, plus a new element x which belongs to no X_i . The topology on X is the smallest topology which contains each open set of each X_i , and which contains $\{x\} \dot{\cup} \bigcup_{i \in E} X_i$, for every $E \subseteq I$ such that $I \setminus E$ is finite.

Proposition 8. *If $(X_i)_{i \in I}$ is a family of topological spaces, then their Frechet disjoint union $X = \{x\} \dot{\cup} \bigcup_{i \in I} X_i$ is T_0 , T_1 , Hausdorff, regular, normal, D -compact (for a given ultrafilter D), $[\lambda, \mu]$ -compact (for given infinite cardinals λ and μ), has a base of clopen sets if and only if so is (has) each X_i .*

Proof. Straightforward. We shall comment only on regularity, normality and D -compactness.

For regularity and normality, just observe that if C is closed in X and C has nonempty intersection with infinitely many X_i 's, then $x \in C$.

As for D -compactness, suppose D is over J and that each X_i is D -compact. Let $(y_j)_{j \in J}$ be a sequence of elements of X . If $\{j \in J \mid y_j \in \bigcup_{i \in F} X_i\} \notin D$ holds for every $F \subseteq I$, then $(y_j)_{j \in J}$ D -converges to x . Otherwise, since D is an ultrafilter, hence ω -complete, there exists some $i \in I$ such that $\{j \in J \mid y_j \in X_i\} \in D$. But then $(y_j)_{j \in J}$ D -converges to some point of X_i , since X_i is supposed to be D -compact. \square

When κ is singular of cofinality ω , Condition (vi) in Theorem 5 is equivalent to the other conditions. When each κ_i is either a regular cardinal, or a singular cardinal of cofinality ω , then Condition (vi) in Theorem 6. is equivalent to the other conditions. Proofs shall be given elsewhere.

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